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ABSTRACT

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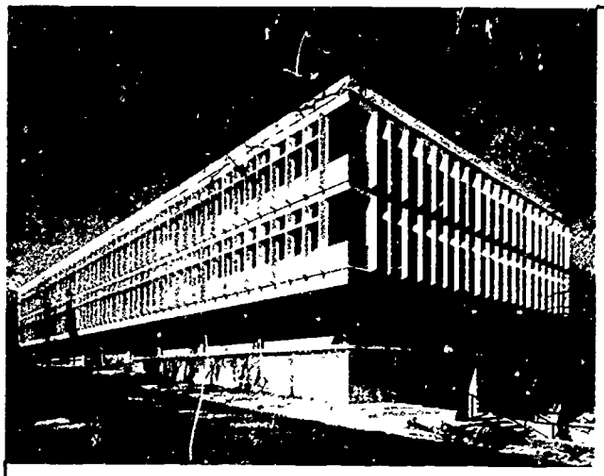

POLYNOMIAL CONJOINT ANALYSIS OF SIMILARITIES: A MODEL FOR CONSTRUCTING POLYNOMIAL CONJOINT MEASUREMENT ALGORITHMS.

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Polynomial conjoint analysis of similarities: A model for constructing polynomial conjoint measurement algorithms.[1]

A model permitting construction of algorithms for the polynomial conjoint analysis of similarities is presented. This model, which is based on concepts used in nonmetric scaling, permits one to obtain the best approximate solution. The concepts used to construct nonmetric scaling algorithms are reviewed. Finally, examples of algorithmic models for nonmetric scaling, multidimensional unfolding, conjoint measurement, factor analysis, subjective expected utility, and the Bradley-Terry-Luce choice problem are presented.

In his paper on polynomial conjoint measurement, Tversky (1967) indicated that one of the important unsolved problems faced by his and similar measurement models is the construction of algorithms for obtaining numerical solutions commensurate with the model. It is the purpose of this paper to indicate a general solution to this problem.

The first section of this paper presents a brief review of the polynomial conjoint measurement model proposed by Tversky. In the next section, it is noted that the problem of algorithm construction has been solved for one polynomial conjoint

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measurement model, the nonmetric multidimensional scaling model. A thorough review of the concepts of nonmetric scaling algorithms is presented, and it is proposed that the same concepts can be successfully adopted for a wide range of polynomial conjoint measurement models. In the next section of the paper a model permitting the construction of algorithms for polynomial conjoint analysis is presented. In the final section several examples of specific submodels are presented.

Polynomial conjoint measurement.

Tversky (1967) noted that one of the goals of scientific investigation may be regarded as the decomposition of complex phenomena into sets of basic factors according to some specified rules of combination. When the factors can be measured independently one desires to account for their joint effects by the appropriate combination rule. It is often the case, however, that the factors cannot be measured independently, and that only the order of their joint effects is known. In this case it is desirable to be able to simultaneously reduce the complex phenomena to its basic factors and to obtain a measurement of these basic factors such that the combination of the factors accounts for the order of the observations. This is the conjoint measurement problem, and the combination rule is known as the conjoint measurement model.

In particular, a data matrix meets the requirements for polynomial conjoint measurement if some monotonic transformation of the data matrix can be decomposed into several factors. The

decomposition rule must be some specified series of sums, differences, and products of the factors themselves. Such a decomposition rule is called a polynomial function.

In his paper, Tversky investigated the necessary and sufficient conditions under which a data matrix can be represented by a polynomial conjoint measurement model. It is not the purpose of this paper to delve into these conditions, but rather to present a method for measuring the factors and their effects, conditions permitting. If the conditions do not permit such measurement, then the method to be presented obtains a least squares estimate of the measurements and their effects, as well as providing information concerning the accuracy of the estimates.

In his paper, Tversky (1967) presents several examples of polynomial conjoint combination rules. These rules include the Hullian and Spencian performance models cited in Hilgard (1965), the Bradley-Terry-Luce choice model (Luce, 1959), the subjective expected utility model (Savage, 1954), and the nonmetric multidimensional scaling models (Coombs, 1964; Shepard, 1964). For one of these models, the nonmetric multidimensional scaling model, the computation problem has been thoroughly investigated (Guttman, 1968) and several computer programs exist (Kruskal, 1964; McGee, 1966; Lingoes, 1965; Young, 1968). The relationship between several of the algorithms has been investigated by Young and Appelbaum (1968).

It is the hypothesis of this paper that the general approach to construction of algorithms for nonmetric multidimensional scaling may also serve as an approach for constructing algorithms

for polynomial conjoint analysis. In fact, the former is a special case of the latter. When the method for constructing algorithms for nonmetric scaling is understood, and when the relation between nonmetric scaling and polynomial conjoint analysis is understood, then it is clear what steps must be taken to generalize nonmetric scaling algorithms to obtain polynomial conjoint analysis algorithms.

Nonmetric scaling algorithms.

In 1962 Shepard introduced the first algorithm for nonmetric multidimensional scaling. He stated that the goal of this analytic method was to derive the metric structure of an unknown configuration of points in a Euclidian space of unknown dimensionality on the basis of nonmetric information about the proximity of the points. That is, Shepard's method attempted to simultaneously convert the proximity measures into Euclidian distances, and to obtain the coordinates underlying the distances. In polynomial conjoint measurement terms, the Shepard method, by applying a Euclidian combination rule, obtained the factors (coordinates) whose effects (Euclidian distances) were monotonic with the proximity measures. In matrix notation, Shepard's developments can be expressed as

$$S \overset{m}{=} D = f(X), \tag{1}$$

where S is the symmetric matrix of proximities between p points, D is the p-order symmetric matrix of Euclidian distances, and X is the rectangular matrix of r-dimensional coordinates with p rows and r columns. The symbol $\overset{m}{=}$ is used to indicate that the matrix D is monotonic with the matrix S. That is, if $s_{ij} > s_{kl}$,

then $d_{ij} \leq d_{kl}$. As indicated in the equation, matrix D is related to the matrix S through the function f. The function is the Euclidian distance function, and is performed on corresponding elements in all pairs of rows of X. The function is defined as

$$f(X) = \left[\sum_{a=1}^{r} (x_{ia}-x_{ja})^2 \right]^{1/2} , \text{ for } i,j=1,..,p \quad . \qquad (2)$$

Notice that for Shepard's developments the monotonicity requirement is actually a weak decreasing monotonicity requirement. That is, his requirement is weak in the sense that two distances may equal each other even though the two corresponding proximities do not, and his requirement is decreasing in the sense that smaller distances correspond with larger proximities.

The analysis of proximities, as represented by equations (1) and (2), served as the basis for the development of a method by Kruskal (1964a; 1964b) which became known as nonmetric multidimensional scaling. Perhaps the most important difference between the two methods is that Kruskal desired to obtain a matrix of distances that was a least squares fit to a matrix representing a monotonic transformation of the similarities. Notice that this differs from the Shepard approach by introducing an objective definition of the best solution. As a by-product of objectifying the definition of the best solution, Kruskal found it necessary to introduce a new matrix. This matrix, called the matrix of disparities by the current author (Young, 1968b), allowed Kruskal to perform computations on numbers which were monotonic with the similarities without actually violating

the ordinal assumptions about the similarities.

A second important difference between the Shepard and Kruskal methods is that Kruskal generalized his definition of the distance function to include all Minkowski spaces. The fimiliar Euclidian space is a special case of the more general Minkowski space, as is the "city-block" space used by Attneave (1950).

In matrix notation, Kurskal's developments can be expressed as

$$S \overset{m}{=} \Delta \cong D = g(X) \tag{3}$$

where Δ is the matrix of disparities (symmetric with $\underline{p}$ rows and columns), and where the symbol $\cong$ indicates a least squares approximation. The matrix D is related to the coordinates X by the function $\underline{g}$. This is the Minkowski distance function and is defined as

$$g(X) = \left[\sum_{a=1}^{r} (x_{ja} - x_{ja})^{c} \right]^{1/c} \text{, for } i, j = 1,2,\ldots,p, \tag{4}$$

where the function is defined for corresponding elements in all pairs of rows of X, and where c is the Minkowski constant such that $c \geq 1$.

In summary, nonmetric scaling, as represented by Kruskal's developments, allowed the analysis of similarities in any Minkowski space, such that the best possible monotonic transformation was obtained. In polynomial conjoint measurement terms, Kruskal's nonmetric scaling, using a combination rule defined by the Minkowski distance function in equation (4), was able to

simultaneously obtain the factors (coordinates) and their effects (Minkowski distances) such that the effects were monotonic with the data matrix (similarities).

Following Kruskal's developments several investigators have introduced analogous methods of analysis (Lingoes, 1965; McGee, 1966; Young, 1968a). An extremely thorough discussion of the general considerations for constructing nonmetric scaling algorithms has been presented by Guttman (1968). The relations among several of the methods have been discussed by Young & Appelbaum (1968).

In the next section of this paper it is shown how equations (3) and (4) can be generalized in order to apply the well understood methods of nonmetric scaling algorithms to polynomial conjoint analysis of similarities.

Polynomial conjoint analysis of similarities.

The model for constructing algorithms for polynomial conjoint analysis of similarities involves two fundamental generalizations of the nonmetric scaling model. One of these generalizations involves modifying the function relating the matrix X to the matrix D, and the other generalization involves removing the restriction that the matrix S be a symmetric matrix.

Analysis of rectangular matrices. The key to understanding the generalization of the method to include rectangular matrices is the concept of a supermatrix. It will prove useful to rewrite equation (3) in supermatrix notation as

$$\left[\begin{array}{c|c} S_{11} & S_{12} \\ \hline S_{21} & S_{22} \end{array}\right] \underset{m}{\cong} \left[\begin{array}{c|c} \Delta_{11} & \Delta_{12} \\ \hline \Delta_{21} & \Delta_{22} \end{array}\right] \underset{\cong}{} \left[\begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & D_{22} \end{array}\right] = g\begin{pmatrix} X_1 \\ -- \\ X_2 \end{pmatrix} \quad . \quad (5)$$

That is, we are re-defining the matrix S of similarities as being a supermatrix, and in a parallel manner are re-defining the matrices Δ, D, and X as being supermatrices.

Consider each submatrix in equation (5). The matrix S_{11} contains the similarities of one set of stimuli, let us say set 1. Notice that the similarities are of the stimuli <u>within</u> set 1. The matrix S_{22} contains parallel information for the stimuli <u>within</u> set 2. Both these matrices are necessarily symmetric. We will denote the number of rows and columns in S_{11} as $\underline{p}_1$, and the number of rows and columns in S_{22} as $\underline{p}_2$. Turning our attention to the matrix S_{12}, we notice that it contains similarities <u>between</u> stimuli in sets 1 and 2. This matrix is rectangular with $\underline{p}_1$ rows and $\underline{p}_2$ columns. We note that S_{21} is simply the transpose of S_{12}. The same relationships hold for the matrices of disparities and distances. In a corresponding manner, the matrix X_1 contains coordinates for the stimuli in set 1 and X_2 for the stimuli in set 2. X_1, therefore, has $\underline{p}_1$ rows and $\underline{r}$ columns, and X_2 has $\underline{p}_2$ rows and $\underline{r}$ columns.

The final step in generalizing the Kruskal model to include the analysis of rectangular matrices as well as symmetric matrices is to assume that there is no information concerning

the similarities within sets 1 and 2. On the basis of this assumption we write

$$S_{12} \overset{m}{=} \Delta_{12} \tilde{=} D_{12} = g' \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{6}$$

where S_{12} is a rectangular matrix of similarities which is monotonic with Δ_{12}, the rectangular matrix of disparities. The matrix Δ_{12} is, in turn, a least squares approximation to D_{12}, the rectangular matrix distances. The distances in D_{12}, in turn, are between the points in set 1 (whose coordinates are represented by X_1) and those in set 2 (whose coordinates are represented by X_2). The definition of the function g is slightly modified so that each row in X_1 is compared with each row in X_2: we denote the new function g' and it is defined as

$$g'(X) = \left[\sum_{a=1}^{r} (x_{ia} - x_{ja})^c \right]^{1/c}, \text{ for } \begin{matrix} i = 1,2,\ldots,p_1 \\ j = 1,2,\ldots,p_2 \end{matrix} . \tag{7}$$

It should be noted that by applying function g as defined by equation (4) to the submatrix X_1 we obtain

$$D_{11} = g(X_1), \tag{8}$$

and applying it to X_2 we obtain

$$D_{22} = g(X_1) . \tag{9}$$

Let us look at matrices X_1 and X_2 for a moment. Both matrices have r columns corresponding with the dimensionality of the space the analysis is being performed in. The number of rows in X_1 corresponds with the number of rows in S_{12}, whereas

the rows of X_2 correspond with the columns of S_{12}. That is, the matrix X_1 may be thought of as representing the row effects of data matrix S_{12}, and X_2 represents the column effects of the data. Note that X_1 and X_2 must be of the same dimensionality and are determined up to a joint unit and rotation.

In summary, the matrices X_1 and X_2 are combined, through the operation $\underline{g}'$, to produce the matrix D_{12}. D_{12}, in turn, is a least squares fit to Δ_{12}, given that Δ_{12} is perfectly monotonic with the data S_{12}. The monotonicity restraint may be either increasing or decreasing and is weak. The matrix D_{11} is related to X_1 by the operation $\underline{g}$, and D_{22} is related to X_2 by the same operation $\underline{g}$.

Generalized function. The second generalization of the nonmetric model is to relax the function relating the matrix D of distances and X of coordinates. The revised function is denoted $\underline{h}$ for symmetric cases, and is defined as

$$h(X) = h_1(h_2(x_{i.},x_{j.})), \text{ for } i, j=1, 2, \ldots,p, \qquad (10)$$

and for rectangular cases is denoted $\underline{h}'$ and is defined as

$$h'(X) = h_1' (h_2'(x_{i.},x_{j.})), \text{ for } \begin{matrix} i=1,2,\ldots,p_1 \\ j=1,2,\ldots,p_2 \end{matrix}, \qquad (11)$$

where the notation $x_{i.}$ is used to indicate the entire i'th row of X.

The entire model for the polynomial conjoint analysis of similarities can be represented, for the symmetric case, by the equation

$$S \overset{m}{=} \Delta \tilde{=} D = h(X), \qquad (12)$$

where h is defined by equation (10). For the nonsymmetric case, the model is represented by the equation

$$S_{12} \overset{m}{=} \Delta_{12} \tilde{=} D_{12} = h' \begin{pmatrix} X_1 \\ \hline X_2 \end{pmatrix} , \tag{13}$$

where h' is defined by equation (11). It should be noted that the function h' can also be applied to X_1 and X_2 in the rectangular case, giving us

$$D_{11} = h'(X_1) ,$$

and

$$D_{22} = h'(X_2) . \tag{14}$$

In summary, for symmetric analyses the matrix X contains the coordinates (or factors or dimensions, etc.) whose distances in the space defined by the function $\underline{h}$ best reproduce the order (or the inverse of the order) of the entries in the data matrix S. For rectangular analyses the matrix X_1 contains the row coordinates (or row effects or row factors, etc.) and the matrix X_2 contains the column coordinates (or column effects, or column factors, etc.) whose between-set distances in the space defined by the function $\underline{h}'$ best reproduce the order (or the inverse of the order) of the entries in the rectangular data matrix S_{12}.

Specific Submodels of the General Model

The function $\underline{h}$ relating the matrices D and X is too general to be of immediate interest. It is possible, however, to make specific assumptions concerning the functions $\underline{h}_1$ and $\underline{h}_2$, generating what will be called specific submodels of the general model.

Some examples of a few familiar submodels are presented below.

Euclidian scaling. The submodel for standard Euclidian nonmetric multidimensional scaling is obtained by assuming

$$\underline{h}_1(\underline{h}_2) = [\underline{h}_2]^{1/2} \quad ,$$

and

$$\underline{h}_2(x_{i.},x_{j.}) = \sum_{a=1}^{r} (x_{ia}-x_{ja})^2 \quad .$$

With these assumptions equation (10) becomes the familiar Euclidian distance function presented earlier as equation (2). This submodel corresponds directly with one of the programs of Guttman and Lingoes (Lingoes, 1965), and with the program presented by McGee (1966).

Minkowski scaling. The submodel for nonmetric multidimensional scaling in any Minkowski space is provided by assuming

$$\underline{h}_1(\underline{h}_2) = [\underline{h}_2]^{1/c} \quad ,$$

and

$$\underline{h}_2(x_{i.},x_{j.}) = \sum_{a=1}^{r} |x_{ia}-x_{ja}|^c \quad ,$$

where c is, as before, the Minkowski constant. With these assumptions equation (10) becomes the Minkowski distance function presented as equation (7). This submodel corresponds directly with the model proposed by Kruskal, and with the program prepared by Young and Torgerson (1967). One of the important Minkowski spaces which has been used in psychological research is the city-block space corresponding with a Minkowski constant of 1. Attneave (1950) has reported some analyses using this space.

Multidimensional unfolding. The rectangular version of the Euclidian nonmetric multidimensional scaling submodel corresponds with the multidimensional unfolding model proposed by Coombs (1964). The submodel is obtained by assuming

$$\underline{h}_1' (\underline{h}_2') = [\underline{h}_2']^{1/2} \quad ,$$

and

$$\underline{h}_2' (x_{i.}, x_{j.}) = \sum_{a=1}^{r} (x_{ia} - x_{ja})^2 \quad .$$

With these assumptions equation (11) becomes a Euclidian distance function between two sets of coordinates. This function is the one proposed by Coombs, and corresponds with the program prepared by Lingoes (1966), and the program written by Young (1968a, 1968b).

Minkowski unfolding. The rectangular version of the Minkowski nonmetric multidimensional scaling submodel generates a model which would logically be called a Minkowski unfolding model. This submodel is obtained by assuming

$$\underline{h}_1' (\underline{h}_2') = [\underline{h}_2']^{1/c} \quad ,$$

and

$$\underline{h}_2' (x_{i.}, x_{j.}) = \sum_{a=1}^{r} |x_{ia} - x_{ja}|^c \quad .$$

The author is unaware of anyone having proposed this model, but the program by Young (1968a, 1968b) is capable of performing analyses based on this model.

Dominance metric. In the area of discrimination and generalization several different models have been presented to account

for response generalization when the stimuli are multidimensional. Some of these models are discussed by Cross (1967) and, as he has pointed out, they correspond with differing Minkowski spaces. One of the models corresponds with the Euclidian scaling model, and another corresponds with the city-block model discussed earlier. A third model, which Cross calls the dominance model, corresponds with a Minkowski space with infinite Minkowski constant. In a dominance space the distance between two points is defined as being equal to the largest of the absolute differences between the coordinates. In the terminology being used here, we would define the dominance submodel as

$$\underline{h}_1(\underline{h}_2) = \underline{h}_2 \quad ,$$

and

$$\underline{h}_2(x_{i.},x_{j.}) = \max_{a=1}^{r} (|x_{ia}-x_{ja}|) \quad ,$$

where the vertical lines indicate absolute value. No computational method has been proposed for this model, to the knowledge of the author. However, with the Kruskal model, several available programs will provide essentially equivalent results by using a very large number for the Minkowski constant.

Conjoint measurement. Luce and Tukey (1964) have presented a powerful measurement model which they refer to as the conjoint measurement model. This is basically an additive model and it can be represented as the specific submodel

$$\underline{h}_1' (\underline{h}_2') = \underline{h}_2' \quad ,$$

and

$$\underline{h}_2' (x_{j.}, x_{j.}) = x_{il} + x_{jl} \quad .$$

Two programs exist which can perform analyses according to this model, one written by Tversky & Zivian (1966), and one by Lingoes (1968).

Polynomial conjoint measurement. A subset of the models proposed by Tversky may be generated from our general model by defining the submodel

$$\underline{h}_1' (\underline{h}_2') = [\underline{h}_2']^b \quad ,$$

and

$$\underline{h}_2' (x_{j.}, x_{j.}) = \sum_{a=1}^{r} (x_{ia} + x_{ja})^c \quad ,$$

where b and c are integer constants. In this case, equation (11) becomes

$$d_{ij} = \left[\sum_{a=1}^{r} (x_{ia} + x_{ja})^c \right]^b \quad . \qquad (15)$$

The submodel represented by equation (15) is actually a class of submodels, with different submodels generated by different sets of assumptions concerning the constants r, b, and c. A few examples follow.

If we assume that r = 1, b = 1, and c = 2, then we see that

$$d_{ij} = x_{il}^2 + 2x_{il}x_{jl} + x_{jl}^2 \quad ,$$

which is simply the quadratic function of two variables. If we assumed that r = 1, b = 1, and c = 3, then we would obtain the formula for the cubic function of two variables. If, on the other hand, we were to assume that r = 2, c = 1, and b = 2, we

would obtain a complex power function of four variables. It should be clear that by the correct selection of the parameters $\underline{b}$ and $\underline{c}$ we can determine the degree of the polynomial under consideration, and that by changing the value of $\underline{r}$ we can modify the number of variables in the equation.

Nonmetric factor analysis. Several nonmetric analogs of factor analysis have been proposed (Shepard 1962; Lingoes, 1967b). One possible analog, differing from those presented earlier, will be presented here. This is specifically an analog of the Tucker and Messick points-of-view model (1963) as discussed by Cliff (1968) and Young and Pennell (1967). If one defines the submodel as

$$\underline{h}_1' \ (\underline{h}_2') = \underline{h}_2'$$

and

$$\underline{h}_2' \ (x_{i.}, x_{j.}) = \sum_{a=1}^{r} (x_{ia} x_{ja}) \quad ,$$

then equation (11) becomes

$$d_{ij} = \sum_{a=1}^{r} (x_{ia} x_{ja}) \quad ,$$

or, in matrix terms,

$$D = X_1 X_2' \quad .$$

This corresponds with the Tucker and Messick model which involves the matrix equation (using our symbols)

$$D = X_1 \Gamma X_2' \quad ,$$

where Γ is a diagonal matrix of weights.

Subjective expected utility. According to the subjective expected utility model (Savage, 1954), when a subject chooses between two gambles he makes his choice by maximizing the subjective expected utility of the choices. The subjective expected utility of a gamble is equal to the sum, over the various choice objects, of the product of the utility of an outcome and its subjective probability of occurance.

For this submodel one defines

$$h_1'(h_2') = h_2' \quad ,$$

and

$$h_2'(x_{i.}, x_{j.}) = \sum_{a=1}^{r} (x_{ia} x_{ja}) \quad ,$$

where there are r outcomes for each gamble, and where the $x_{i.}$ represent the utilities and the $x_{j.}$ the subjective probabilities. It should be obvious that the nonmetric analog of the factor analysis model and the subjective expected utility model are formally identical.

Bradley-Terry-Luce choice model. This model (Luce, 1959) specifies the relation of choice probabilities when two choice objects are presented to the scale values of the objects. The model states that

$$p(c,d) = \frac{v(c)}{v(c) + v(d)}$$

where v(c) represents the scale values. The ordinal version of this model can be written

$$p(c,d) < p(e,f) \iff v(c) - v(d) < v(e) - v(f) \quad .$$

In the terminology used here, if we assume

$$h_1'(h_2') = h_2' \quad ,$$

and

$$h_2'(x_{i.}, x_{j.}) = x_{i1} - x_{j1} \quad ,$$

then

$$d_{ij} = x_{i1} - x_{j1} \quad ,$$

where d_{ij} takes on the role of the choice probability between objects i and j, and x_{i1} and x_{j1} are the scale values of those objects. It should be noticed that this is equivalent to a one dimensional Minkowski metric.

Conclusions

On the basis of notions fundamental to nonmetric multidimensional scaling, a model has been developed which indicates a method for constructing algorithms for the polynomial conjoint analysis of similarities. It has been shown that this model includes, as special submodels, several of the common forms of nonmetric scaling, many of the forms of polynomial conjoint analysis, and several popular choice models. It should be obvious that, with the proper specification of the functional relationships indicated by equation (10) or (11), a great range of polynomial conjoint models is possible.

Perhaps one of the major advantages of the model presented here is that it provides a means for minimizing the complex functions represented by equations (10) through (13). The iterative minimization algorithms used in nonmetric scaling may be applicable to this new model. In a subsequent paper, an

objective definition of what is meant by a "best solution" will be presented, along with a definition of a combination rule including a wide range of useful polynomial conjoint measurement submodels.

References

Attneave, F. Dimensions of similarity. American Journal of Psychology, 1950, 63, 516-556.

Cliff, N. The "idealized individual" interpretation of individual differences in multidimentional scaling, Psychometrika, 1968, 33, 225-232.

Coombs, C. H., A theory of data. New York: Wiley 1964.

Cross, D. V. Metric properties of multidimansional stimulus generalization. In David I. Mostofsky (Ed.), Stimulus generalization. Stanford: Stanford University Press, 1965,

Guttman, L. A general nonmetric technique for finding the smallest coordinate space for a configuration of points. Psychometrika, 1968, 33, 469-506.

Hilgard, E. R. Theories of learning. New York: Appleton-Century-Croft, 1956.

Kruskal, J. B. Multidimensional scaling by optimizing goodness-of-fit to a nonmetric hypothesis. Psychometrika, 1964a, 29, 1-28.

Kruskal, J. B. Nonmetric multidimensional scaling: A numerical method. Psychometrika, 1964b, 29, 115-130.

Lingoes, J. C. An IBM-7090 program for Guttman-Lingoes smallest space analysis - I. Behavioral Science, 1965, 10, 183-184.

Lingoes, J. C. An IBM-7090 program for Guttman-Lingoes smallest space analysis - RI. Behavioral Science, 1966, 11, 332.

Lingoes, J. C. An IBM 360/67 program for Guttman-Lingoes conjoint measurement - III. Behavioral Science, 1968, 13, 421-422.

Luce, R. D. Individual choice behavior. New York: Wiley, 1959.

Luce, R. D. & Tukey, J. W. Simultaneous conjoint measurement: Journal of Mathematical Ps chology, 1964, 1, 1-27.

McGee, V. E. The multidimensional analysis of 'elastic' distances. The British Journal of Mathematical and Statistical Psychology, 1966, 19, 181-196.

Savage, L. J. The foundation of statistics. New York: Wiley, 1954.

Shepard, R. N. The analysis of proximities: Multidimensional scaling with an unknown distance function: II. Psychometrika, 1962, 27, 125-140.

Shepard, R. N. Metric structures in ordinal data. Journal of Mathematical Psychology, 1966, 3, 287-315.

Tucker, L. R. & Messick, S. An individual differences model for multidimensional scaling. Psychometrika, 1963, 28, 333-367.

Tversky, A. A general theory of polynomial conjoint measurement. Journal of Mathematical Psychology, 1967, 4, 1-20.

Tversky, A. & Zivian, A. An IBM 7090 program for additivity analysis. Behavioral Science, 1966, 11, 78-79.

Young, F. W. TORSCA-9: A FORTRAN IV program for nonmetric multidimensional scaling. Behavioral Science, 1968a, 13, 343-344.

Young. F. W. A FORTRAN IV program for nonmetric multidimensional scaling. Chapel Hill, North Carolina: L. L. Thurstone Psychometric Laborator' Report No. 56, 1968b.

Young, F. W. & Pennell, R. An IBM system/360 program for points of view analysis. Behavioral Science, 1967, 12, 166.

Young, F. W. & Torgerson, W. S. TORSCA, A FORTRAN IV program for Shepard-Kruskal multidimensional scaling analysis. Behavioral Science, 1967, 12, 498.